

IMPORTANT CALENDAR CHANGES

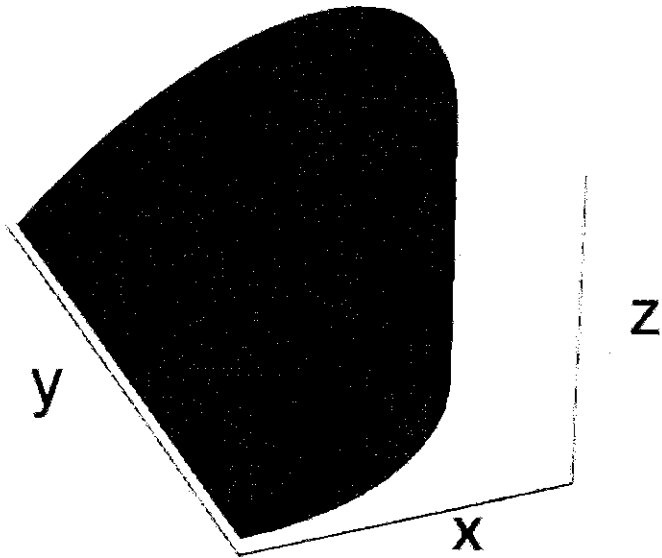
Close *Tuesday*: 15.3, 15.4

Exam 2 is THURSDAY!!!

10.3, 13.4, 14.1, 14.3, 14.4, 14.7, 15.1-4

Entry Task (Old Exam Question)

Find the volume of the wedge shaped solid that lies above the xy -plane, below the plane $z = x$, and within the solid cylinder $x^2 + y^2 \leq 9$.



$$\iint_D x \, dA$$

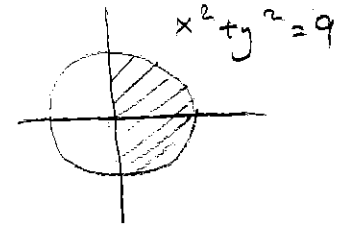
$$\int_{-\pi/2}^{\pi/2} \int_0^3 r \cos \theta \, r \, dr \, d\theta$$

$$\int_{-\pi/2}^{\pi/2} \cos \theta \left. \frac{1}{3} r^3 \right|_0^3 d\theta$$

$$= 9 \int_{-\pi/2}^{\pi/2} \cos \theta \, d\theta$$

$$= 9 \left(\sin \theta \Big|_{-\pi/2}^{\pi/2} \right)$$

$$= 9 (1 - -1) = \boxed{18}$$



15.4 Center of Mass

New App: Consider a thin plate (*lamina*) with density at each point given by

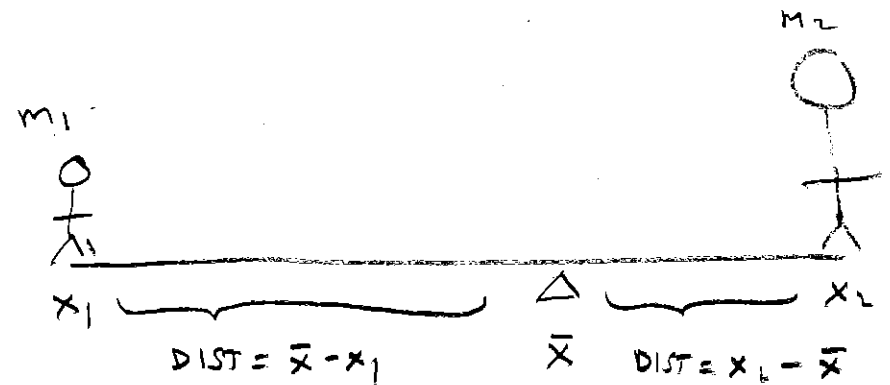
$$\rho(x, y) = \text{mass/area (kg/m}^2\text{)}.$$

We will see that the center of mass (centroid) is given by

$$\begin{aligned} \bar{x} &= \frac{\text{"Moment about y"}}{\text{Total Mass}} \\ &= \frac{\iint_R x \rho(x, y) dA}{\iint_R \rho(x, y) dA} \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{\text{"Moment about x"}}{\text{Total Mass}} \\ &= \frac{\iint_R y \rho(x, y) dA}{\iint_R \rho(x, y) dA} \end{aligned}$$

Motivation "the see-saw"

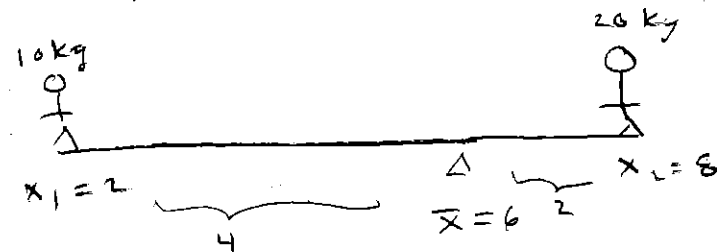


$$\begin{aligned} \text{LAW OF LEVEL} &\Rightarrow m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x}) \\ &\Rightarrow m_1\bar{x} - m_1x_1 = m_2x_2 - m_2\bar{x} \\ &\Rightarrow m_1\bar{x} + m_2\bar{x} = m_1x_1 + m_2x_2 \\ &\Rightarrow (m_1 + m_2)\bar{x} = m_1x_1 + m_2x_2 \\ &\bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \end{aligned}$$

EX $m_1 = 10, m_2 = 20$
 $x_1 = 2, x_2 = 8$

$$\bar{x} = \frac{(10)(2) + (20)(8)}{10 + 20} = \frac{20 + 160}{30} = \frac{180}{30} = 6$$

"MOMENT ABOUT y-AXIS"
"TOTAL MASS"



In general: If you are given n points
 $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with
 corresponding masses m_1, m_2, \dots, m_n
 then

$$\bar{x} = \frac{m_1 x_1 + \dots + m_n x_n}{m_1 + \dots + m_n} = \frac{M_y}{M}$$

$$\bar{y} = \frac{m_1 y_1 + \dots + m_n y_n}{m_1 + \dots + m_n} = \frac{M_x}{M}$$

Derivation:

1. Break region into m rows and n columns.
2. Find center of mass of each rectangle:

$$(\bar{x}_{ij}, \bar{y}_{ij})$$

3. Estimate the mass of each rectangle:

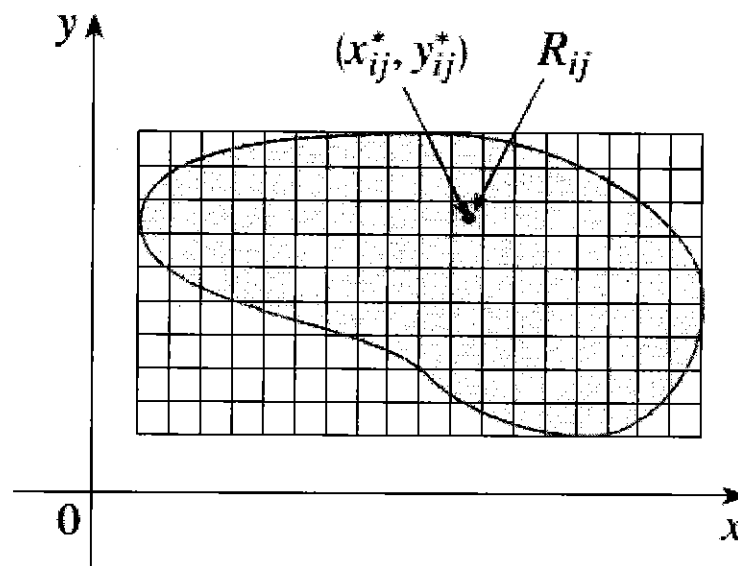
$$m_{ij} = p(\bar{x}_{ij}, \bar{y}_{ij}) \Delta A$$

4. Now use the formula for n points.

5. Take the limit.

$$\bar{x} = \frac{\sum_{i=1}^m \sum_{j=1}^n m_{ij} x_{ij}}{\sum_{i=1}^m \sum_{j=1}^n m_{ij}}$$

$$= \frac{\sum_{i=1}^m \sum_{j=1}^n \bar{x}_{ij} p(\bar{x}_{ij}, \bar{y}_{ij}) \Delta A}{\sum_{i=1}^m \sum_{j=1}^n p(\bar{x}_{ij}, \bar{y}_{ij}) \Delta A}$$



Center of Mass:

$$\bar{x} = \frac{\text{Moment about } y}{\text{Total Mass}} = \frac{\iint_R x p(x, y) dA}{\iint_R p(x, y) dA}$$

$$\bar{y} = \frac{\text{Moment about } x}{\text{Total Mass}} = \frac{\iint_R y p(x, y) dA}{\iint_R p(x, y) dA}$$

Example:

Consider a 1 by 1 m square metal plate.
The density is given by $p(x,y) = kx$ kg/m²
for some constant k .
Find the center of mass.

$$\begin{aligned} \text{TOTAL MASS} &= \int_0^1 \int_0^1 kx \, dx \, dy = \int_0^1 \left. \frac{k}{2} x^2 \right|_0^1 dy \\ &= \frac{k}{2} \int_0^1 1 \, dy = \left. \frac{k}{2} y \right|_0^1 = \boxed{\frac{k}{2} = M} \end{aligned}$$

$$\begin{aligned} \text{MOMENT ABOUT } y &= \int_0^1 \int_0^1 x (kx) \, dx \, dy = \int_0^1 \left. \frac{k}{3} x^3 \right|_0^1 dy \\ &= \frac{k}{3} \int_0^1 1 \, dy = \boxed{\frac{k}{3} = M_y} \end{aligned}$$

$$\begin{aligned} \text{MOMENT ABOUT } x &= \int_0^1 \int_0^1 y (kx) \, dx \, dy = \int_0^1 \left. \frac{k}{2} y x^2 \right|_0^1 dy \\ &= \frac{k}{2} \int_0^1 y \, dy = \left. \frac{k}{2} \cdot \frac{1}{2} y^2 \right|_0^1 = \boxed{\frac{k}{4} = M_x} \end{aligned}$$

$$\bar{x} = \frac{M_y}{M} = \frac{k/3}{k/2} = \frac{2}{3}$$

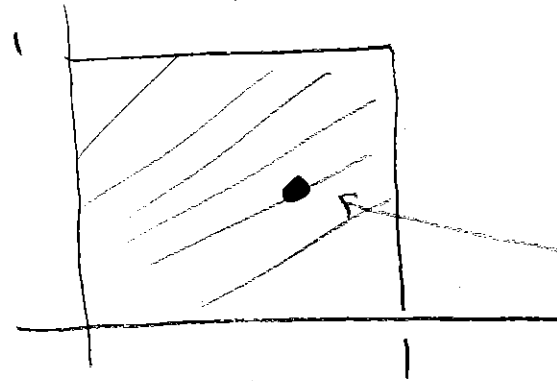
$$\bar{y} = \frac{M_x}{M} = \frac{k/4}{k/2} = \frac{1}{2}$$

CENTER OF MASS

$$\left(\frac{2}{3}, \frac{1}{2} \right)$$

Side note:

The density $p(x,y) = kx$ means that the density is proportional to x which can be thought of as distance from the y -axis. In other words, the plate gets heavier at a constant rate from left-to-right.



Example) If $k=15$
then

$$\begin{aligned} p\left(\frac{1}{3}, 1\right) &= 15 \cdot \frac{1}{3} = 5 & \frac{k}{3} \\ p\left(\frac{2}{3}, 1\right) &= 15 \cdot \frac{2}{3} = 10 & \frac{2k}{3} \\ p(1, 1) &= 15 \cdot 1 = 15 & k \end{aligned}$$

SAME ANSWER NO MATTER WHAT k IS.

Translations:

Density proportional to the dist. from...

...the y-axis -- $p(x, y) = kx$.

...the x-axis -- $p(x, y) = ky$.

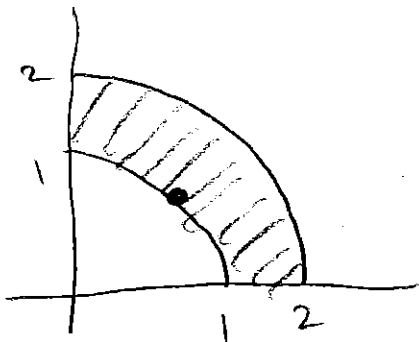
...the origin -- $p(x, y) = k\sqrt{x^2 + y^2}$.

Density proportional to the square of the distance from the origin:

$$p(x, y) = k(x^2 + y^2).$$

Density inversely proportional to the distance from the origin:

$$p(x, y) = \frac{k}{\sqrt{x^2 + y^2}}$$



Example: A thin plate is in the shape of the region bounded between the circles of radius 1 and 2 in the first quadrant.

The density is proportional to the distance from the origin.

Find the center of mass.

$$p(x, y) = k\sqrt{x^2 + y^2}$$

$$\begin{aligned}
 M &= \iint_D p(x, y) dA = \int_0^{\pi/2} \int_1^2 kr \cdot r dr d\theta \\
 &= \int_0^{\pi/2} \frac{k}{2} r^2 \Big|_1^2 d\theta \\
 &= \int_0^{\pi/2} \frac{k}{2} (8-1) d\theta \\
 &= \frac{7k}{2} \theta \Big|_0^{\pi/2} = \frac{7\pi}{6} k
 \end{aligned}$$

$$\begin{aligned}
 M_y &= \iint_D x p(x, y) dA = \int_0^{\pi/2} \int_1^2 r \cos\theta \cdot kr \cdot r dr d\theta \\
 &= k \int_0^{\pi/2} \cos\theta \cdot \frac{1}{4} r^4 \Big|_1^2 d\theta = \frac{k}{4} (16-1) \sin\theta \Big|_0^{\pi/2} \\
 &= \frac{15k}{4}
 \end{aligned}$$

$$\begin{aligned}
 M_x &= \iint_D y p(x, y) dA = \int_0^{\pi/2} \int_1^2 r \sin\theta \cdot kr \cdot r dr d\theta \\
 &= k \int_0^{\pi/2} \sin\theta \cdot \frac{1}{4} r^4 \Big|_1^2 d\theta = \frac{k}{4} (16-1) (-\cos\theta \Big|_0^{\pi/2}) = \frac{15k}{4}
 \end{aligned}$$

$$\begin{aligned}
 \bar{x} &= \frac{15k/4}{7\pi k/6} = \frac{45}{14\pi} \\
 &\approx 1.02313
 \end{aligned}$$

$$\begin{aligned}
 \bar{y} &= \frac{15k/4}{7\pi k/6} = \frac{45}{14\pi} \\
 &\approx 1.02313
 \end{aligned}$$

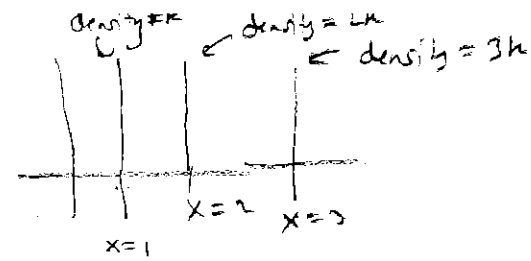
Translations:

Density proportional to the dist. from...

...the y-axis -- $p(x, y) = kx$.

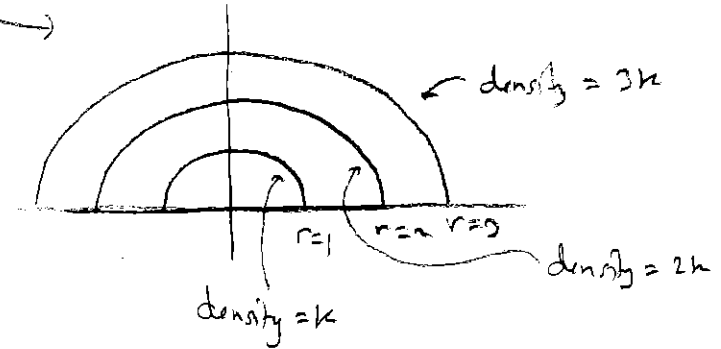
...the x-axis -- $p(x, y) = ky$.

...the origin -- $p(x, y) = k\sqrt{x^2 + y^2}$.



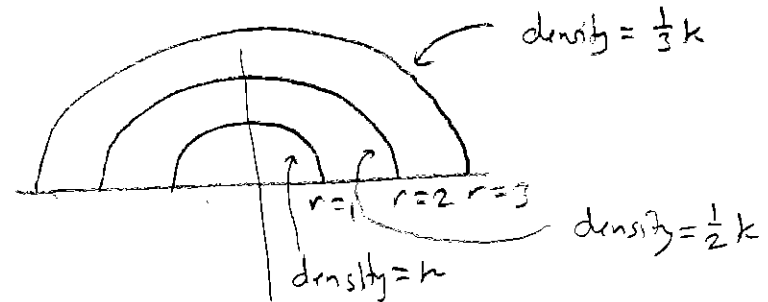
Density proportional to the square of the distance from the origin:

$$p(x, y) = k(x^2 + y^2).$$



Density inversely proportional to the distance from the origin:

$$p(x, y) = \frac{k}{\sqrt{x^2 + y^2}}$$

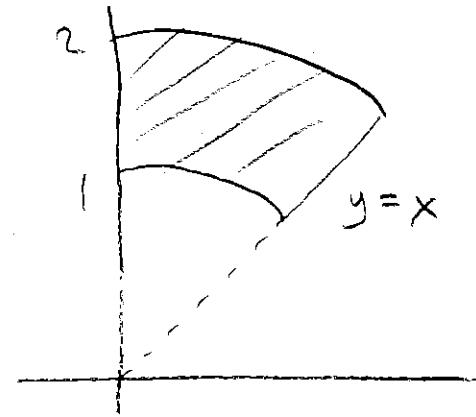


Example (Old Exam Question)

A lamina occupies the region R in the first quadrant that is above the line $y = x$ and between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

The density is proportional to the distance from the origin.

Find the y -coordinate of the center of mass.



$$\rho(x,y) = k\sqrt{x^2+y^2}$$

$$\bar{y} = \frac{\iint_D y \rho(x,y) dA \leftarrow \text{I}}{\iint_D \rho(x,y) dA \leftarrow \text{II}}$$

$$\begin{aligned} \text{I} &= \int_{\pi/4}^{\pi/2} \int_1^2 r \sin \theta \cdot k r \cdot r dr d\theta \\ &= k \int_{\pi/4}^{\pi/2} \sin \theta \int_1^2 r^3 dr d\theta \\ &= k \int_{\pi/4}^{\pi/2} \sin \theta \left. \frac{1}{4} r^4 \right|_1^2 d\theta \\ &= \frac{k}{4} (16-1) \int_{\pi/4}^{\pi/2} \sin \theta d\theta \\ &= \frac{15k}{4} (-\cos \theta) \Big|_{\pi/4}^{\pi/2} \\ &= \frac{15k}{4} (0 - -\frac{\sqrt{2}}{2}) = \frac{15\sqrt{2}k}{8} \end{aligned}$$

$$\begin{aligned} \text{II} &= \int_{\pi/4}^{\pi/2} \int_1^2 k r \cdot r dr d\theta \\ &= k \int_{\pi/4}^{\pi/2} \left. \frac{1}{3} r^3 \right|_1^2 d\theta \\ &= \frac{k}{3} (8-1) \int_{\pi/4}^{\pi/2} 1 d\theta \\ &= \frac{7k}{3} \theta \Big|_{\pi/4}^{\pi/2} \\ &= \frac{7k}{3} (\pi/2 - \pi/4) \\ &= \frac{7\pi k}{12} \end{aligned}$$

$$\bar{y} = \frac{\text{I}}{\text{II}} = \frac{7\pi/12}{15\sqrt{2}/8} = \frac{7\pi/3}{15\sqrt{2}/2} = \frac{14\pi}{45\sqrt{2}}$$

$$\approx 1.446936$$